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Angular Momentum, Magnetic Moment, and *g*-Factor in General Relativity

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The solutions to the Einstein-Maxwell equations for the case of a slowly rotating, massive thin shell with arbitrary charge are investigated. The form of the metric chosen here facilitates a more detailed analysis of the shell's angular momentum, magnetic moment, and g-factor than in earlier work. In addition to confirming earlier results, it is found that, for a charge-to-mass ratio greater than unity there is no upper or lower bound on the value g may take and that the magnetic moment and net angular momentum of the shell may vanish or change sign (relative to the sense of rotation).

1. INTRODUCTION

In an earlier paper (Briggs *et al.*, 1981) the Einstein-Maxwell equations were solved for the case of a slowly rotating body with arbitrary charge. The solution was then applied to an infinitesimally thin, rotating, charged spherical shell, and the angular momentum and g-factor of the shell were calculated.

For such a calculation, the stationary axially symmetric metric (Brill and Cohen, 1966; Cohen and Brill, 1968) used to describe the space-time is of the form

$$ds^2 = -A^2\,dt^2 + B^2\,dr^2 + C^2\,d\theta^2 + E^2(d\phi - \Omega\,dt)^2 \tag{1.1}$$

where A, B, C, E, and Ω are functions of r and θ. The angular velocity of inertial frames, Ω, is measured relative to inertial frames of the asymptotically flat space-time at infinity.

In this paper we follow in part the outline of the previous work (Briggs *et al.*, 1981), but rather than writing the metric in isotropic form in equation (1.1), we choose to use the standard Schwarzschild representation for a

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spherical mass and charge distribution (see Section 2). This has the advantage of simplifying the calculation somewhat.

We obtain expressions for the g-factor, angular momentum, and magnetic moment as functions of the charge-to-mass and radius-to-mass ratios, and comment on these. We also include several graphs to illustrate the dependence of these quantities on intermediate as well as limiting values of the charge-to-mass and radius-to-mass ratios.

2. EINSTEIN–MAXWELL EQUATIONS: FORMULATION AND SOLUTION

The Einstein field equations are

$$8\pi T^{\mu\nu} = G^{\mu\nu}$$

where $T^{\mu\nu}$ and $G^{\mu\nu}$ are components of the stress-energy tensor and the Einstein tensor, respectively. A suitable dual basis in which to perform the calculation is $\omega^0 = dt$, $\omega^1 = B\,dr$, $\omega^2 = C\,d\theta$, $\omega^3 = E(d\phi - \Omega\,dt)$. The nontrivial field equations [as stated in Briggs *et al.* (1981)] are

$$-8\pi T^{00} = (BC)^{-1}\left[\left(\frac{C_r}{B}\right)_r + \left(\frac{B_\theta}{C}\right)_\theta + E^{-1}\left(\frac{CE_r}{B}\right)_r + E^{-1}\left(\frac{BE_\theta}{C}\right)_\theta\right] + \left(\frac{E\Omega_r}{2AB}\right)^2 + \left(\frac{E\Omega_\theta}{2AC}\right)^2$$

$$-8\pi T^{03} = \tfrac{1}{2}(BCE^2)^{-1}\left[\left(\frac{CE^3\Omega_r}{AB}\right)_r + \left(\frac{BE^3\Omega_\theta}{AC}\right)_\theta\right]$$

$$-8\pi T^{12} = (AC)^{-1}\left[\left(\frac{A_r}{B}\right)_\theta - \frac{A_\theta C_r}{BC}\right] + (CE)^{-1}\left[\left(\frac{E_r}{B}\right)_\theta - \frac{E_\theta C_r}{BC}\right] - \frac{E^2\Omega_r\Omega_\theta}{2A^2BC}$$

$$8\pi T^{11} = \left(\frac{E\Omega_r}{2AB}\right)^2 - \left(\frac{E\Omega_\theta}{2AC}\right)^2 + \frac{E_rC_r}{CB^2E} + \frac{A_rE_r}{AB^2E} + \frac{C_rA_r}{AB^2C} + (CE)^{-1}\left(\frac{E_\theta}{C}\right)_\theta + (AC)^{-1}\left(\frac{A_\theta}{C}\right)_\theta + \frac{A_\theta E_\theta}{AC^2E}$$

$$8\pi T^{22} = \left(\frac{E\Omega_\theta}{2AC}\right)^2 - \left(\frac{E\Omega_r}{2AB}\right)^2 + \frac{E_\theta B_\theta}{BC^2E} + \frac{A_\theta B_\theta}{ABC^2} + \frac{A_\theta E_\theta}{AC^2E} + (AB)^{-1}\left(\frac{A_r}{B}\right)_r + (BE)^{-1}\left(\frac{E_r}{B}\right)_r + \frac{A_rE_r}{AB^2E}$$

$$8\pi T^{33} = (ABC)^{-1}\left[\left(\frac{CA_r}{B}\right)_r + \left(\frac{BA_\theta}{C}\right)_\theta\right] + (BC)^{-1}\left(\frac{C_r}{B}\right)_r + (BC)^{-1}\left(\frac{B_\theta}{C}\right)_\theta - 3\left(\frac{E\Omega_r}{2AB}\right)^2 - 3\left(\frac{E\Omega_\theta}{2AC}\right)^2$$

where the subscript r or θ denotes partial differentiation.

The mechanical contributions to the stress-energy tensor for a rigid body that is observed to rotate slowly with angular velocity ω about the z axis is given (to first order in ω and Ω) by

$$T^{\mu\nu}_{\text{mech}} = \begin{pmatrix} \rho_m & \xi t^{31} & \xi t^{32} & \xi(\rho_m + t^{33}) \\ \xi t^{13} & t^{11} & t^{12} & t^{13} \\ \xi t^{23} & t^{21} & t^{22} & t^{23} \\ \xi(\rho_m + t^{33}) & t^{31} & t^{32} & t^{33} \end{pmatrix} + \text{higher order terms}$$

where ρ_m is the mass density, $\xi \equiv (E/A)(\omega - \Omega)$, and $t^{\mu\nu}$ is the mechanical stress-energy tensor in a frame that is not rotating with respect to the observer. We note that in such a frame $t^{0i} = 0$ $(i = 1, 2, 3)$.

The electromagnetic contribution to the stress-energy tensor is[3]

$$8\pi T^{00}_{\text{em}} = E^2 + H^2$$
$$8\pi T^{0i}_{\text{em}} = 2\varepsilon^{i}_{jk} e^j h^k$$
$$8\pi T^{ij}_{\text{em}} = (E^2 + H^2)\delta^{ij} - 2(e^i e^j + h^i h^j)$$

The electric and magnetic fields are

$$\mathbf{E} = e^1\omega_1 + e^2\omega_2 + e^3\omega_3$$
$$\mathbf{H} = h^1\omega_1 + h^2\omega_2 + h^3\omega_3$$

The dual representation of Maxwell's equations gives the following expressions (correct to first order in ω and Ω):

$$(ACe_2)_{,r} - (ABe_1)_{,\theta} - BCE\left[h_2\left(\frac{\Omega_\theta}{C}\right) + h_1\left(\frac{\Omega_r}{B}\right)\right] = 0$$
$$(AEe_3)_{,r} = 0$$
$$(AEe_3)_{,\theta} = 0$$
$$(BEh_2)_{,\theta} + (CEh_1)_{,r} = 0$$

and

$$(ACh_2)_{,r} - (ABh_1)_{,\theta} + BCE\left[e_2\left(\frac{\Omega_\theta}{C}\right) + e_1\left(\frac{\Omega_r}{B}\right)\right] = 4\pi\rho_e\xi ABC$$
$$(AEh_3)_{,r} = 0$$
$$(AEh_3)_{,\theta} = 0$$
$$(CEe_1)_{,r} + (BEe_2)_{,\theta} = 4\pi\rho_e BCE$$

where ρ_e is the local charge density.

[3]For a definition of the symmetric electromagnetic stress-energy tensor see, for example, Jackson (1975).

We now find an exterior solution to the Einstein–Maxwell equations using the metric in standard Schwarzschild form:

$$ds^2 = -A^2\,dt^2 + B^2\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,(d\phi - \Omega\,dt)^2$$

where A and B are functions only of r. The electric and magnetic fields may be written as

$$\mathbf{E} = e\omega_1, \qquad \mathbf{H} = n\cos\theta\,\omega_1 + p\sin\theta\,\omega_2$$

where e, p, and n are functions only of r. Neglecting terms quadratic in the angular velocity and noting that $\mathbf{H}$ is linear in the angular velocity, the nontrivial field equations are those involving T^{00}, T^{03}, T^{11}, and T^{22} (or equivalently T^{33}). They are

$$8\pi r^2\left(\rho_m + \frac{e^2}{8\pi}\right) = \left[r\left(1 - \frac{1}{B^2}\right)\right]_{,r} \tag{2.1}$$

$$2Aep + 8\pi(\omega - \Omega)(\rho_m + t^{33})r = -\frac{A}{2Br^3}\left(\frac{r^4\Omega_{,r}}{AB}\right)_{,r} \tag{2.2}$$

$$A^2B^2\left[1 + 8\pi r^2\left(t^{11} - \frac{e^2}{8\pi}\right)\right] = (rA^2)_{,r} \tag{2.3}$$

$$8\pi AB\left(t^{33} + \frac{e^2}{8\pi}\right) = \left[\frac{(rA)_{,r}}{rB}\right]_{,r} + \frac{A}{Br^2} \tag{2.4}$$

The only nontrivial Maxwell equations are the fourth, fifth, and eighth. They are

$$p = -\frac{1}{2Br}(r^2 n)_{,r} \tag{2.5}$$

$$(Arp)_{,r} + ABn + r^2 e\Omega_{,r} = 4\pi\rho_e r^2 B(\omega - \Omega) \tag{2.6}$$

$$(r^2 e)_{,r} = 4\pi\rho_e B r^2 \tag{2.7}$$

For the remainder of this section we shall consider the region exterior to the charge and mass distributions. In this region $\rho_m = \rho_e = t^{\mu\nu} = 0$. A first integral of equation (2.7) is

$$e(r) = q/r^2 \tag{2.8}$$

where q is a constant to be determined. With this expression for $e(r)$, equation (2.1) reduces to $B^2 = (1 - k/r + q^2/r^2)^{-1}$, where k is a constant to be determined. After some manipulation equation (2.3) yields $A^2 = k_1(1 - k/r + q^2/r^2)$, where k_1 is a constant to be determined. Since the space is asymptotically flat at infinity, we choose $A^2 \to 1$ as $r \to \infty$. This requires $k_1 = 1$. For large r, we thus have $A^2 \sim 1 - k/r + O(r^{-2})$. This must match onto the weak field solution, which has $A^2 = 1 - 2m/r$. This requires $k = 2m$. Hence,

$$A^2 = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$

$$B^2 = A^{-2} \tag{2.9}$$

in the region exterior to the charge and mass distributions. We may now proceed to solve $\Omega(r)$ and $n(r)$. Substituting equation (2.5) into (2.2) we obtain

$$q(nr^2)_{,r} = \frac{1}{2}\left(\frac{r^4\Omega_{,r}}{AB}\right)_{,r}$$

A first integral is

$$\Omega_{,r} = \frac{1}{r^4}(2qnr^2 - \eta_0) \tag{2.10}$$

where η_0 is a constant to be determined, and we have chosen $AB = +1$. Using equations (2.10) and (2.5) to eliminate p and $\Omega_{,r}$ in equation (2.6), we obtain

$$-\frac{1}{2}\left[\frac{A}{B}(r^2 n)_{,r}\right]_{,r} + n\left(1 + \frac{2q^2}{r^2}\right) = \frac{q}{r^4}\eta_0 \tag{2.11}$$

or, substituting for A and B,

$$\frac{3q^2}{r^2}n - n'\left(2r - 3m + \frac{q^2}{r}\right) - \tfrac{1}{2}n''(r^2 - 2mr + q^2) = \frac{q}{r^4}\eta_0 \tag{2.12}$$

The particular solution may be found by power series. This method also yields one linearly independent solution of the homogeneous equation. To find the second linearly independent solution of the homogeneous equation, one may write it as the product of the first solution with an (as yet) undetermined function f and then use equation (2.12) to obtain a first-order differential equation for f, which may be solved by the method of partial fractions. Further details may be found in the Appendix of Briggs *et al.* (1981).

We give here the general solution for $|q| \neq m$, which has essentially the same structure as the solution that arises when the isotropic form of the metric is used. For $|q| \neq m$ the general solution of equation (2.12) is

$$n(r) = \frac{\eta_0 q}{3mr^3} + \eta_1\left(1 - 3\zeta^2\frac{m^2}{r^2} + 2\zeta^4\frac{m^3}{r^3}\right) + \eta_2\left[4\zeta^2\frac{m^2}{r^2} - \frac{2m}{r} - \frac{2}{3}(1+2\zeta^2)\zeta^{-2}\right.$$
$$\left. - \frac{1}{b}\left(1 - 3\zeta^2\frac{m^2}{r^2} + 2\zeta^4\frac{m^3}{r^3}\right)\ln\frac{r - m(1+b)}{r - m(1-b)}\right] \tag{2.13}$$

where $\zeta \equiv |q|/m$ and $b \equiv (1-\zeta^2)^{1/2}$. Note that for $|q| > m$,

$$b^{-1}\ln\frac{r-m(1+b)}{r-m(1-b)} = |b|^{-1}\tan^{-1}\frac{2|b|(r/m-1)}{|b|^2 - (r/m-1)^2}$$

since the argument of the logarithm is of the form $z/\bar{z}$ and thus the modulus of the argument is unity. Integration of equation (2.10) using (2.12) yields Ω. For $|q| \neq m$

$$\Omega(r) = \Omega_0 + \frac{\eta_0}{3r^3}\left(1 - \zeta^2\frac{m}{2r}\right) - \frac{2q\eta_1}{r}\left(1 - \zeta^2\frac{m^2}{r^2} + \zeta^4\frac{m^3}{2r^3}\right)$$
$$-2q\eta_2\left[\zeta^2\frac{m^2}{r^3} - \frac{m}{r^2} - \frac{2\zeta^{-2}}{3r}(1+\tfrac{1}{2}\zeta^2)\right.$$
$$\left. - \frac{1}{2mb}\left(\frac{2m}{r} - 2\zeta^2\frac{m^3}{r^3} + \zeta^4\frac{m^4}{r^4} - 1\right)\ln\frac{r-m(1+b)}{r-m(1-b)}\right] \tag{2.14}$$

3. BOUNDARY CONDITIONS

The case of a massive, charged, slowly rotating shell of radius r_0 is considered. The shell rotates rigidly about the z axis with angular velocity ω. Let the mass and charge distributions be specified by

$$\rho_m = \kappa\delta(r - r_0), \qquad \rho_e = \sigma\delta(r - r_0)$$

where $1 = \int \delta(r - r_0)\, d^3r$. Since e must be regular at the origin, equation (2.7) requires $e = 0$ for $r < r_0$. Integration of equation (2.7) across the shell yields

$$q = \int \sigma\delta(r - r_0)\omega^1 \wedge \omega^2 \wedge \omega^3$$

This identifies q as the total charge.

The left-hand side of equation (2.4) contains terms proportional to $\theta(r - r_0)$ ($e^2/8\pi$ term) and $\delta(r - r_0)$ (t^{33} term), where θ is the Heaviside function and δ is the Dirac delta function. One therefore concludes that A is continuous at $r = r_0$. Since t^{11} is zero both inside and outside the shell, equation (2.3) implies that $t^{11} = 0$ everywhere. Regularity of A and B at the origin requires A and B to be constant in the interior, $r < r_0$. Equation (2.3) yields $B = 1$ for $r < r_0$. Thus

$$B = 1, \qquad A^2 = A_0^2 \equiv 1 - \frac{2m}{r_0} + \frac{q^2}{r_0^2} \qquad \text{for} \quad r < r_0 \tag{3.1}$$

Integration of equation (2.1) across $r = r_0$ yields

$$\kappa = m - q^2/2r_0$$

Multiplying both sides of equation (2.1) by B and integrating across $r = r_0$, we obtain

$$8\pi r_0^2 \kappa \int_-^+ B\,\delta(r - r_0)\,dr = -2r_0(A_0 - 1)$$

Note, however, that, in general, the integral of the product of a discontinuous function and a δ-function at the discontinuity is not defined.[4] Let $t^{22} = t^{33} = S\delta(r - r_0)$; if we integrate (2.4) across $r = r_0$ and use the above results, we obtain

$$S = \frac{1}{4A_0}\left[m(1 - A_0) - \frac{q^2}{r_0}\right]$$

Thus,

$$\kappa = m - \frac{q^2}{2r_0}, \qquad S = \frac{1}{4A_0}\left[m(1 - A_0) - \frac{q^2}{r_0}\right]$$

and

$$t^{11} = 0, \qquad t^{22} = t^{33} = S\,\delta(r - r_0) \tag{3.2}$$

Solving equations (2.2), (2.5), and (2.6), we find

$$\Omega = \Omega_0 + \Omega_1/r^3$$
$$n = n_0 + n_1/r^3$$
$$p = -(r^2 n)_{,r}/2r \qquad \text{for} \quad r < r_0$$

Since n and Ω are regular at the origin, $\Omega = \Omega_0$, $n = n_0$, and $p = -n_0$. Outside the shell, Ω is required to vanish at infinity, since Ω is measured relative to observers in inertial frames of the asymptotically flat space-time at infinity. Further, the magnetic fields, since they are physical, must vanish at infinity. These conditions determine η_2 in terms of η_1. We find that

$$n(r) = \frac{\eta_0 q}{3mr^3} + \eta_1 R(r) \tag{3.3}$$

[4]The value of $\int_{-\infty}^{\infty} \theta\delta\,dr$ is undetermined without further information, but lies between 0 and 1; see, for example, the paper by Cohen and Cohen (1971).

where

$$R(r) = -\tfrac{3}{4}m^{-3}(1-\zeta^2)^{-2}\left[\frac{m}{r}+\frac{m^2}{r^2}-\frac{2}{3}\zeta^2(1+2\zeta^2)\frac{m^3}{r^3}\right.$$
$$\left.+\frac{1}{2b}\left(1-3\zeta^2\frac{m^2}{r^2}+2\zeta^4\frac{m^3}{r^3}\right)\ln\frac{r-m(1+b)}{r-m(1-b)}\right]$$

and

$$R(r) = \frac{1}{r^3} + O(r^{-4}) \qquad \text{as} \quad r \to \infty$$

Also

$$\Omega(r) = \frac{\eta_0}{3r^3}\left(1-\zeta^2\frac{m}{2r}\right) + 2q\eta_1 Q(r) \tag{3.4}$$

where

$$Q(r) = -\tfrac{3}{8}m^{-3}(1-\zeta^2)^{-2}\left[\frac{1}{r}-\frac{m}{r^2}-\frac{2}{3}\left(1+\frac{\zeta^2}{2}\right)\frac{m^2}{r^3}+\frac{1}{3}\zeta^2(1+2\zeta^2)\frac{m^3}{r^4}\right.$$
$$\left.-\frac{1}{2mb}\left(\frac{2m}{r}-2\zeta^2\frac{m^3}{r^3}+\zeta^4\frac{m^4}{r^4}-1\right)\ln\frac{r-m(1+b)}{r-m(1-b)}\right]$$

and

$$Q(r) = -\frac{1}{4r^4} + O(r^{-5}) \qquad \text{as} \quad r \to \infty$$

Observe that these solutions have the same structure as those quoted in Briggs *et al.* (1981).

If we assume that p is (at worst) discontinuous at $r = r_0$, then equations (2.2), (2.5), and (2.6) imply that Ω and n are continuous across the shell:

$$\Omega_0 = \frac{\eta_0}{3r_0^3}\left(1-\zeta^2\frac{m}{2r_0}\right) + 2q\eta_1 Q_0 \tag{3.5}$$

and

$$n_0 = \frac{\eta_0 q}{3mr_0^3} + \eta_1 R_0 \tag{3.6}$$

where the subscript indicates that the functions are to be evaluated at $r = r_0$. Integrating equation (2.2) across $r = r_0$ gives

$$\Omega_{,r}\Big|_-^+ = \frac{4}{r_0}(\omega - \Omega_0)(\kappa + S)\frac{A_0 - 1}{A_0\kappa}$$

Integrating equation (2.6) across $r=r_0$ [using equation (2.5)] gives

$$\left[\frac{A}{B}(r^2 n)_r\right]_-^+ = -2q(\omega-\Omega_0)$$

Expanding the above equations, we obtain

$$\frac{\eta_0}{r_0^4}\left(\frac{2\zeta^2 m}{3r_0}-1\right)+2q\eta_1 Q_0' = 4(A_0\kappa r_0)^{-1}(\omega-\Omega_0)(\kappa+S)(A_0-1)$$

or

$$\frac{\eta_0}{r_0^4}\left(\frac{2\zeta^2 m}{3r_0}-1\right)+\frac{2q}{r_0^2}\eta_1 R_0 = 4(A_0\kappa r_0)^{-1}(\omega-\Omega_0)(\kappa+S)(A_0-1) \quad (3.7)$$

[since $Q'(r)=r^{-2}R(r)$], and

$$-\frac{\eta_0 q A_0}{3mr_0^2}(A_0+2)+\eta_1 r_0 A_0[A_0(r_0R_0'+2R_0)-2R_0] = -2q(\omega-\Omega_0) \quad (3.8)$$

where R_0' is the derivative of R with respect to r, evaluated at r_0. We find

$$\eta_1 = \lambda q\eta_0/6m \quad (3.9)$$

where (after some manipulation)

$$\begin{aligned}\lambda &= 2r_0^{-3}[(r_0-\zeta^2 m)+A_0(2r_0-\tfrac{3}{2}m\zeta^2)]\\ &\quad\times\{R_0[(r_0-m\zeta^2)(3A_0^2-1)-2r_0A_0]\\ &\quad+r_0R_0'A_0[(\tfrac{3}{2}r_0-m\zeta^2)A_0+\tfrac{1}{2}(r_0-m\zeta^2)]\}^{-1} \quad (3.10)\end{aligned}$$

and equations (3.7) and (3.8) have been used with (3.1) and (3.2). Before obtaining an expression for the g-factor of the shell, we must first establish a connection between the total angular momentum of the shell J and known constants. (This will also be used in the following section to obtain an explicit representation for J.) The angular momentum is given by[5]

$$J = \int_{t\ \text{constant}} ET^{03}\omega^1\wedge\omega^2\wedge\omega^3 \quad (3.11)$$

with

$$T^{03} = (E/A)(\omega-\Omega)(\rho_m+t^{33})+(1/4\pi)ep\sin\theta$$

[5]For a derivation of Eq. (3.11) see Cohen (1968).

Substituting for E, e, and p [using equations (2.8) and (2.5) for the latter two], we find

$$J = 2\pi \int_0^\pi \sin^3\theta \, d\theta \left[\int_0^\infty (Br^4/A)(\omega-\Omega)(\kappa+S)\,\delta(r-r_0)\,dr - (q/8\pi)\int_{r_0}^\infty (r^2 n)_r\,dr \right]$$

$$= \tfrac{1}{3}[(2r_0^3/\kappa A_0)(\omega-\Omega_0)(\kappa+S)(1-A_0) + q r_0^2 n_0]$$

Eliminating the terms in n_0 and $(\omega-\Omega)$, using equations (3.6) and (3.7), respectively, we find

$$J = \eta_0/6 \tag{3.12}$$

Hence, by equations (3.3) and (3.9),

$$n = \frac{qJ}{mr^3}(2+\lambda) \qquad \text{as} \quad r\to\infty \tag{3.13}$$

The magnetic dipole moment μ is given by

$$n \sim 2\mu/r^3 \qquad \text{as} \quad r\to\infty \tag{3.14}$$

The g-factor is defined by (Panofsky and Phillips, 1962)

$$g \equiv 2\mu(qJ/m)^{-1} \tag{3.15}$$

Hence we find

$$g = 2+\lambda \tag{3.16}$$

4. ANGULAR MOMENTUM AND MAGNETIC MOMENT

In the previous section we obtained a general expression for the g-factor for the case of a massive, charged, slowly rotating shell. The g-factor, however, is not an observable quantity; it depends on the ratio of two physical observables: angular momentum and magnetic moment. In order to complete our discussion of the shell, expressions for the angular momentum J and the magnetic moment μ should be found. In this section we obtain such expressions.

We first proceed to eliminate Ω_0 from equations (3.7) and (3.8) [using equation (3.5)]. After some simplification, equations (3.7) and (3.8) may be written

$$\eta_0 r_0 C_1 + q\eta_1 C_2 = 4\omega(\kappa+S)r_0^3 \tag{4.1}$$

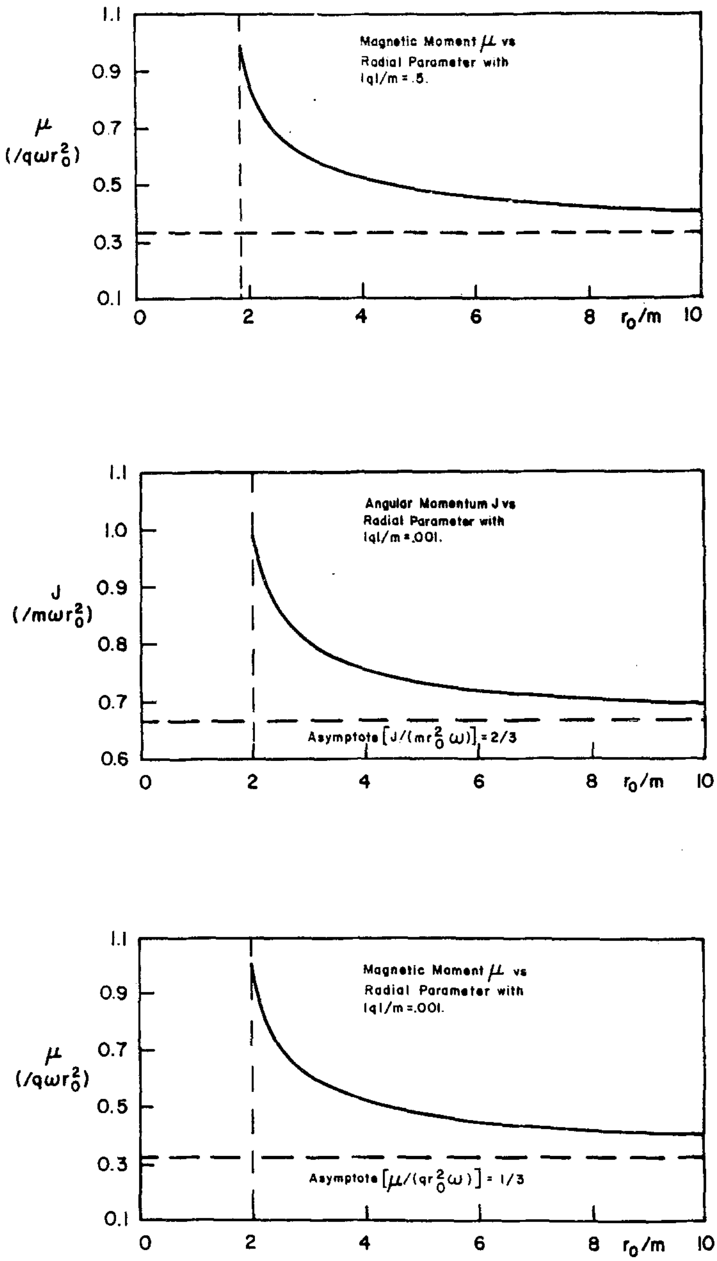

Fig. 1. (a) The g-factor, (b) the angular momentum J in units $m\omega r_0^2$, and (c) the magnetic moment μ in units $q\omega r_0^2$, versus radial parameter r_0/m for $|q|/m = 0.001$.

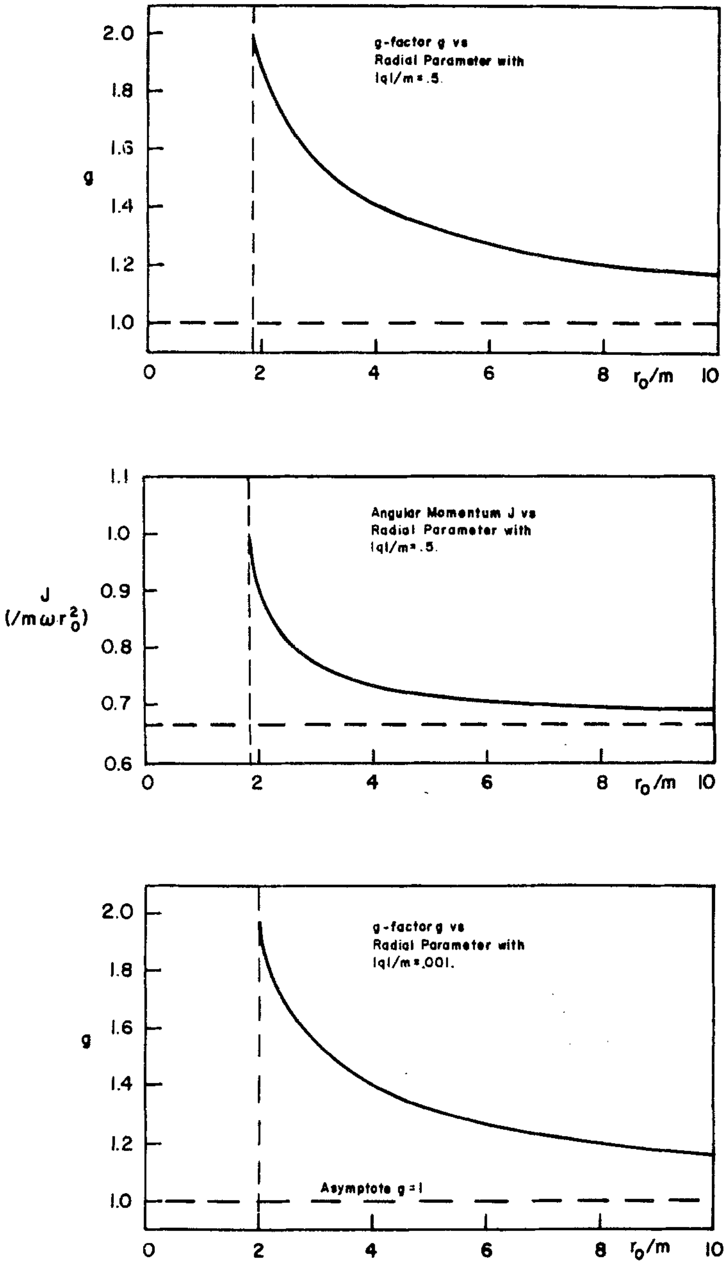

Fig. 2. (a) The g-factor, (b) the angular momentum J in units $m\omega r_0^2$, and (c) the magnetic moment μ in units $q\omega r_0^2$, versus radial parameter r_0/m for $|q|/m = 0.5$.

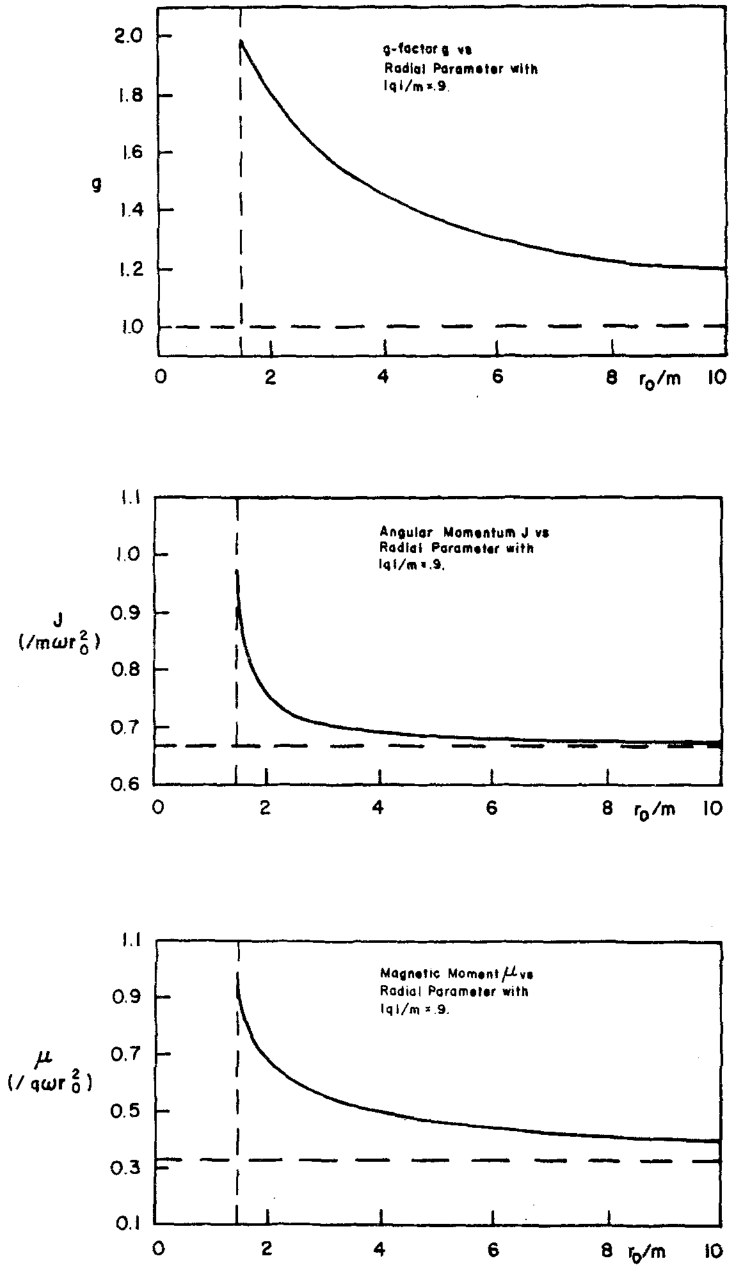

Fig. 3. (a) The g-factor, (b) the angular momentum J in units $m\omega r_0^2$, and (c) the magnetic moment μ in unit $q\omega r_0^2$, versus radial parameter r_0/m for $|q|/m = 0.9$.

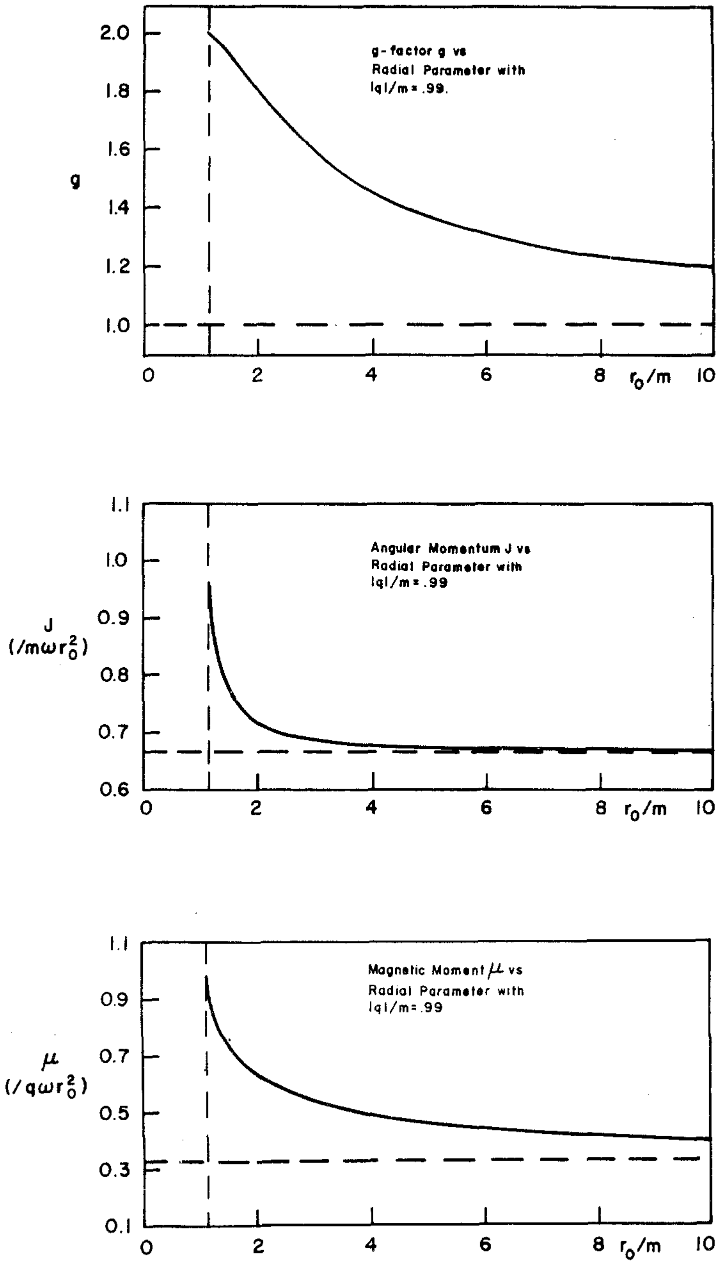

Fig. 4. (a) The g-factor, (b) the angular momentum J in units $m\omega r_0^2$, and (c) the magnetic moment μ in units $q\omega r_0^2$, versus radial parameter r_0/m for $|q|/m = 0.99$.

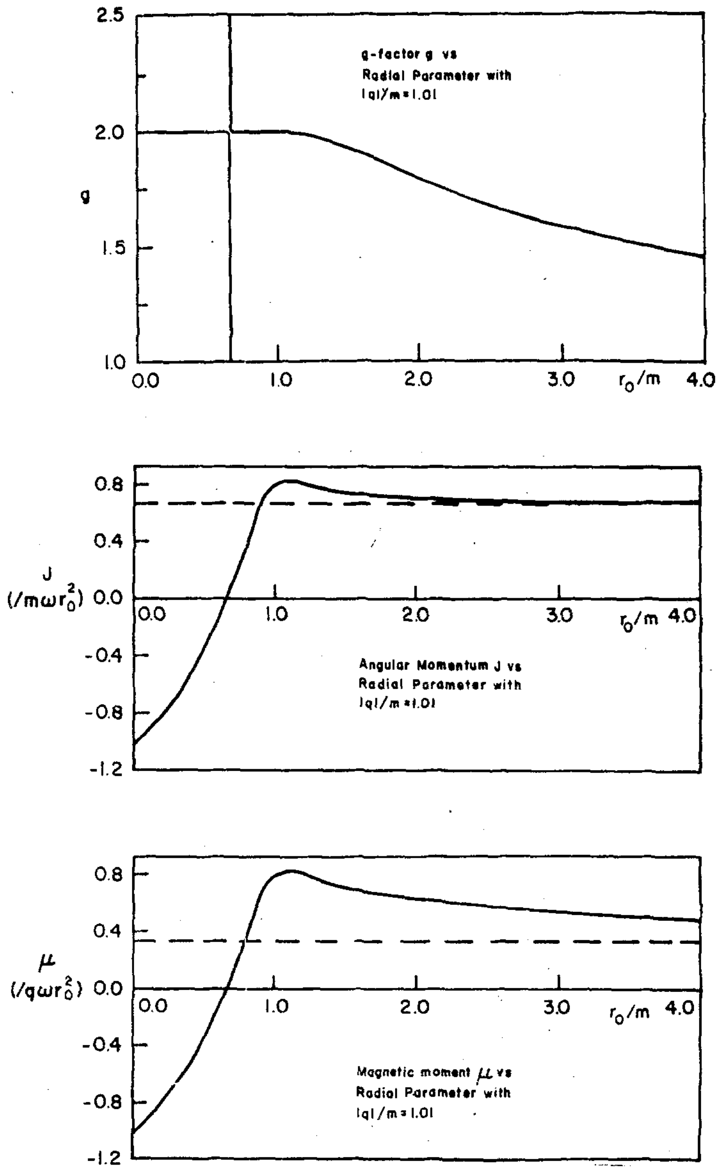

Fig. 5. (a) The g-factor versus radial parameter r_0/m for $|q|/m = 1.01$. Note that g diverges at $r_0/m \sim 0.65$. (b) Angular momentum J in units $m\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 1.01$. Note that J has a root at $r_0/m \sim 0.65$. This almost coincides with the point at which μ has a root. (c) Magnetic moment μ in units $q\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 1.01$. Note that μ has a root at $r_0/m \sim 0.65$. This almost coincides with the point at which J has a root.

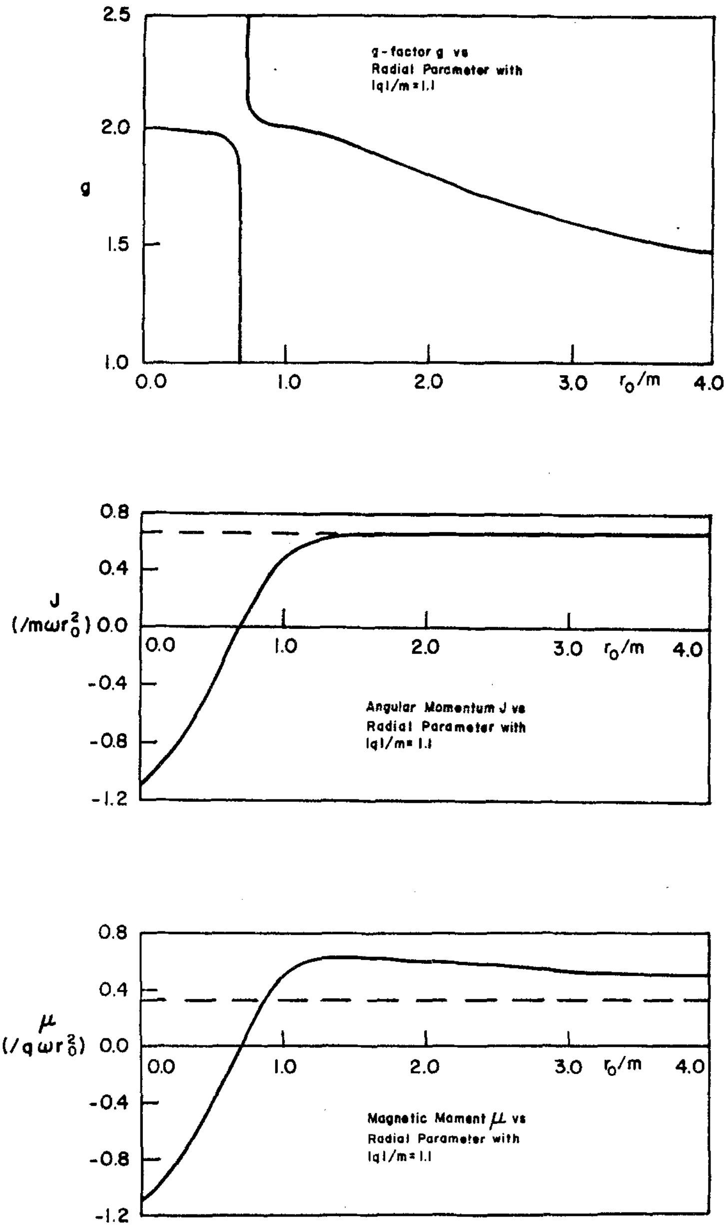

Fig. 6. (a) The g-factor versus radial parameter r_0/m for $|q|/m = 1.1$. Note that g diverges at $r_0/m \sim 0.7$. (b) Angular momentum J in units $m\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 1.1$ Note that J has a root at $r_0/m \sim 0.7$. (c) Magnetic moment μ in units $q\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 1.1$. Note that μ has a root at $r_0/m \sim 0.7$.

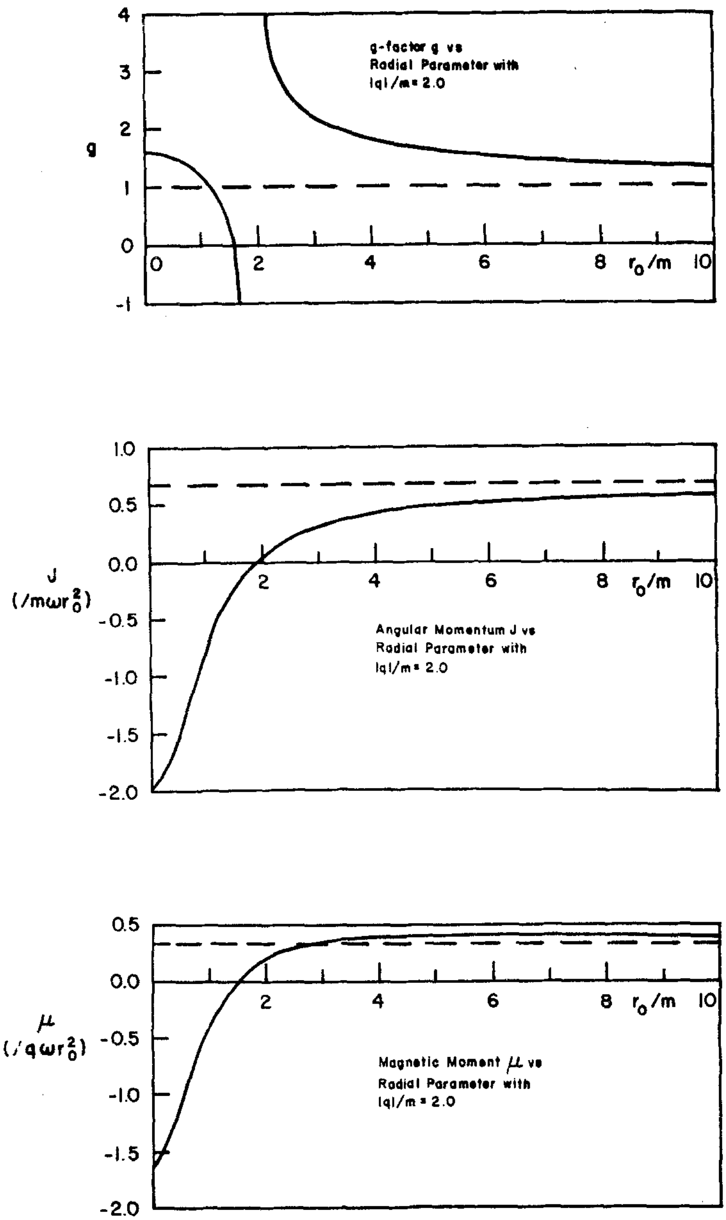

Fig. 7. (a) The g-factor versus radial parameter r_0/m for $|q|/m = 2.0$. Note that g diverges at $r_0/m \sim 1.9$. (b) Angular momentum J in units $m\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 2.0$. Note that J has a root at $r_0/m \sim 1.9$. (c) Magnetic moment μ in units $q\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 2.0$. Note that μ has a root at $r_0/m \sim 1.5$.

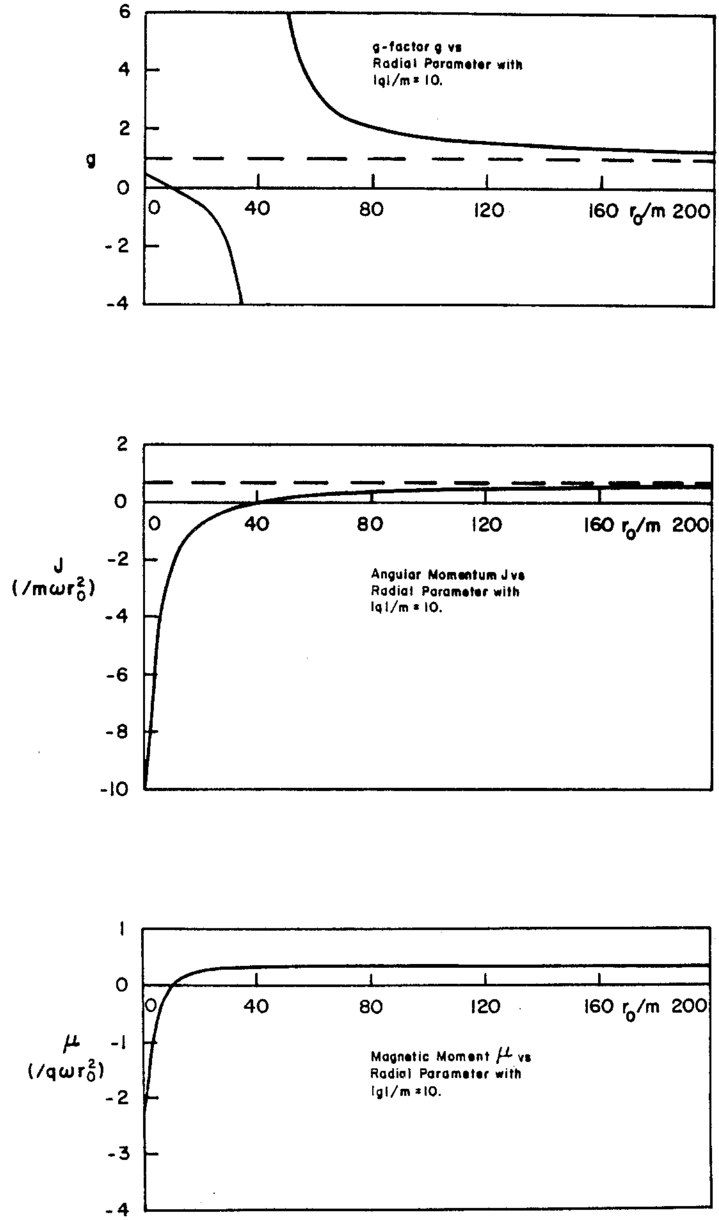

Fig. 8. (a) The g-factor versus radial parameter r_0/m for $|q|/m = 10$. Note that g diverges at $r_0/m \sim 42$. (b) Angular momentum J in units $m\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 10$. Note that J has a root at $r_0/m \sim 42$. (c) Magnetic moment μ in units $q\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 10$. Note that μ has a root at $r_0/m \sim 10$.

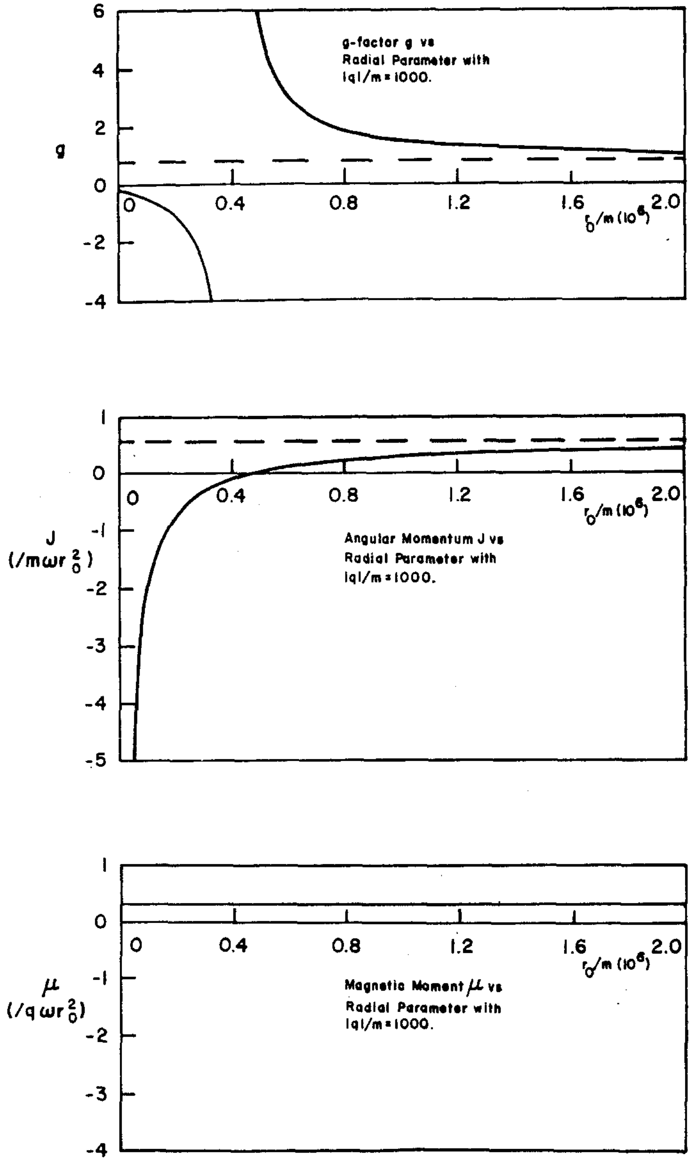

Fig. 9. (a) The g-factor versus radial parameter r_0/m for $|q|/m = 1{,}000$. Note that g diverges at $r_0/m \sim 4.2 \times 10^5$. For $|q|/m \gg 1$ the graph is essentially the same shape as the above. (b) Angular momentum J in units $m\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 1{,}000$. Note that J has a root at $r_0/m \sim 4.2 \times 10^5$. (c) Magnetic moment μ in units $q\omega r_0^2$ versus radial parameter r_0/m for $|q|/m = 1{,}000$. Note that μ has a root at $r_0/m \sim 10^3$, and μ converges very rapidly to its asymptotic value of $q\omega r_0^2/3$.

and

$$(\eta_0 q/6r_0)C_3 - \eta_1 C_4 = 2q\omega r_0^2 \tag{4.2}$$

where the constants C_1, C_2, C_3, and C_4 are defined as follows:

$$\begin{aligned}
C_1 &\equiv \frac{A_0}{2}(1+A_0)\left(1-\frac{4}{3}\zeta^2 x\right)+\frac{4}{3r_0}(1-\zeta^2 x)(\kappa+S)\\
C_2 &\equiv 8(r_0^3 Q_0)(\kappa+S)-A_0(1+A_0)(r_0^3 R_0)\\
C_3 &\equiv \frac{A_0}{x}(A_0+2)+4(1-\zeta^2 x)\\
C_4 &\equiv A_0\{A_0[(r_0^4 R_0')+2(r_0^3 R_0)]-2(r_0^3 R_0)\}-16\zeta^2(r_0^4 Q_0)x^2
\end{aligned} \tag{4.3}$$

and $x \equiv m/2r_0$. Equations (4.1) and (4.2) can be solved simultaneously to determine η_0 and η_1:

$$\eta_0 = 2m\omega r_0^2 \frac{(\kappa+S)C_4 r_0^{-1}+2\zeta^2 x^2 C_2}{xC_1C_4+\frac{2}{3}x^3\zeta^2 C_2 C_3} \tag{4.4}$$

and

$$\eta_1 = 2\omega q r_0^2 \frac{(\kappa+S)C_3 r_0^{-1}-3C_1}{2\zeta^2 C_2 C_3 x^2+3C_1C_4} \tag{4.5}$$

Given equation (4.4), equation (3.12) yields an explicit expression for J. From equations (3.9), (3.13), and (3.14) we find

$$\mu = \tfrac{1}{2}(\tfrac{1}{3}\eta_0 q/m+\eta_1) \tag{4.6}$$

Substituting for η_0 and η_1 [using equations (4.4) and (4.5)] into equation (4.6), one obtains an explicit expression for μ.

For given values of $\zeta = |q|/m$, graphical plots of μ and J as functions of r_0/m (Figures 1-9) illustrate that μ and J are always positive (with respect to the sense of ω) for $\zeta < 1$, but may be either positive or negative for $\zeta > 1$. This will be further discussed in Section 6.

5. ASYMPTOTIC SOLUTIONS FOR $|q| \gg m$ AND $|q| \ll m$

The expressions derived in the previous sections for μ, J, and g are, in general, complicated functions of q, m, r_0, and ω. These expressions do simplify, however, in several limiting cases. In this section we compute the asymptotic formulas for μ, J, and g in these limits. Use will be made of these results in the next section.

5.1. $|q| \gg m$, i.e., $\zeta \gg 1$

In addition, a further requirement—either (i) $|q| \ll r_0$ or (ii) $r_0 \ll m$—is needed in order that the expressions simplify. [The former case, with $\kappa \geq 0$, is non-general relativistic and is the subject of a recent paper (Cohen and Mustafa, 1986).]

(i) $|q|/r_0 \ll 1$, $m/r_0 \ll 1$ (with $\kappa \geq 0$). For this case, equation (3.3) approaches

$$n(r) = \frac{\eta_0}{3r^3}\frac{q}{m} + \frac{\eta_1}{r^3}, \qquad r > r_0 \tag{5.1}$$

while equation (3.4) becomes

$$\Omega(r) = \frac{\eta_0}{3r^3}\left(1 - \frac{m}{2r}\zeta^2\right) - \frac{q\eta_1}{2r^4}, \qquad r > r_0 \tag{5.2}$$

The boundary conditions at the shell, $r = r_0$ [equations (3.5)-(3.8)], become

$$\Omega_0 = \frac{\eta_0}{3r_0^3}(1 - \zeta^2 x) - \frac{q\eta_1}{2r_0^4} \tag{5.3}$$

$$n_0 = \frac{\eta_0}{3r_0^3}\frac{q}{m} + \frac{\eta_1}{r_0^3} \tag{5.4}$$

$$\frac{\eta_0}{r_0^4}(\tfrac{4}{3}\zeta^2 x - 1) + \frac{2q\eta_1}{r_0^5} \sim -\frac{4}{r_0^2}(\omega - \Omega_0)(\kappa + S) \tag{5.5}$$

$$\frac{1}{r_0^2}\left(\eta_0\frac{q}{m} + 3\eta_1\right) \sim 2q(\omega - \Omega_0) \tag{5.6}$$

where $(A_0 - 1)/A_0\kappa \sim -1/r_0$ has been used in equation (5.5). Using equation (5.3) to eliminate Ω_0 from equations (5.5) and (5.6), we find

$$\eta_0(\tfrac{4}{3}\zeta^2 x - 1) + (2q/r_0)\eta_1 \sim -4\omega r_0^3(2x - 3x^2\zeta^2) \tag{5.7}$$

and

$$\eta_0(q/m) + 3\eta_1 \sim 2q\omega r_0^2 \tag{5.8}$$

where use has been made of $(\kappa + S)/r_0 \sim 2x - 3x^2\zeta^2$, and higher order terms in m/r_0 and q/r_0 have been neglected. Equations (5.7) and (5.8) may be solved simultaneously to determine η_0 and η_1:

$$\eta_0 = 2m\omega r_0^2(2 - \tfrac{5}{3}x\zeta^2) \tag{5.9}$$

and

$$\eta_1 = \tfrac{1}{3}\omega q r_0^2(\tfrac{10}{3}\zeta^2 x - 2) \tag{5.10}$$

Equations (3.12) and (5.9) yield

$$J = \tfrac{1}{3} m\omega r_0^2 (2 - \tfrac{5}{3} x\zeta^2) \tag{5.11}$$

while equations (4.6), (5.9), and (5.10) yield

$$\mu = \tfrac{1}{3} q\omega r_0^2 \tag{5.12}$$

The g-factor is found from (3.15) [using equations (5.11) and (5.12)] to be

$$g = 6/(6 - 5\zeta^2 x) \tag{5.13}$$

where $\zeta = |q|/m$ and $x = m/(2r_0)$. This result is in agreement with Cohen and Mustafa (1986) and the corresponding asymptotic result of Briggs *et al.* (1981).

(ii) $r_0/m \ll 1$, $r_0/|q| \ll 1$ (the point particle limit). For this case we use the formulas of Section 4 to obtain the asymptotic expressions for J, μ, and hence g. In this limit equations (4.3) become

$$C_1 = \frac{1}{6}\frac{|q|}{r_0}\left(-\zeta\frac{|q|}{r_0} - 2\zeta + 3\right) + O(r_0^0)$$

$$C_2 = \left(1 + \frac{3\pi}{4\zeta} + \frac{4}{\zeta^2}\right)\left(-\frac{|q|}{2r_0} - 1 + \frac{3}{2\zeta}\right) + O(r_0) + O(\zeta^{-3})$$

$$C_3 = 2\left(2\zeta + \frac{r_0}{m} - 2\frac{r_0}{|q|}\right) + O(r_0^2)$$

$$C_4 = \left(1 + \frac{3\pi}{4\zeta} + \frac{4}{\zeta^2}\right)\left(-2\frac{|q|}{r_0} - 1 + \frac{5}{\zeta}\right) + O(r_0) + O(\zeta^{-3}) \tag{5.14}$$

where use has been made of

$$R_0 = \left(1 + \frac{3\pi}{4\zeta} + \frac{4}{\zeta^2}\right)\left(\frac{1}{r_0^3} - \frac{3}{2mr_0^2\zeta^2}\right) + O(r_0^0) + O(\zeta^{-3})$$

$$Q_0 = \left(1 + \frac{3\pi}{4\zeta} + \frac{4}{\zeta^2}\right)\left(-\frac{1}{4r_0^4} + \frac{1}{2mr_0^3\zeta^2}\right) + O(r_0^{-1}) + O(\zeta^{-3})$$

$$A_0 = \frac{|q|}{r_0} - \frac{1}{\zeta} + O(r_0)$$

$$(\kappa + S)r_0^{-1} = \frac{3m}{4r_0} - \frac{|q|}{4r_0} - \frac{1}{2}\left(\frac{q}{r_0}\right)^2 + O(r_0)$$

Equations (4.4), (4.5) with (5.14) yield

$$\eta_0 = -6m\omega r_0^2[\zeta + O(\zeta^{-2})] + O(r_0^3) \tag{5.15}$$

and

$$\eta_1 = 2\omega q r_0^2\left[\zeta - \frac{3\pi}{4} + O(\zeta^{-1})\right] + O(r_0^3) \tag{5.16}$$

From equations (3.12) and (5.15) we find (for $\zeta \gg 1$)

$$J = -m\omega r_0^2\zeta + O(r_0^3) \tag{5.17}$$

while equations (4.6), (5.15), and (5.16) give (for $\zeta \gg 1$)

$$\mu = -(3\pi/4)\omega q r_0^2 + O(r_0^3) \tag{5.18}$$

We observe from equations (5.17) and (5.18) that $\mu \to 0$ and $J \to 0$ as $r_0 \to 0$. The g-factor is found from equation (3.15) [using equations (5.17) and (5.18)]. It should be noted, however, that (for $\zeta \gg 1$)

$$\lim_{r_0 \to 0} g = 3\pi/2\zeta \tag{5.19}$$

a small, but nonzero, limit.

5.2. $|q| \ll m$, i.e., $\zeta \ll 1$

For this case, equation (3.3) reduces to

$$R(r) = -\tfrac{3}{4}m^{-3}\left[\frac{m}{r} + \frac{m^2}{r^2} + \frac{1}{2}\ln\left(1 - \frac{2m}{r}\right)\right], \qquad r > r_0 \tag{5.20}$$

Note that $qR(r)$ and $qQ(r)$ are $O(\zeta)$. In writing down the boundary conditions at the shell, $r = r_0$, we shall neglect quadratic and higher order terms in ζ in order to obtain equations linear in ζ. Thus, equations (3.5)-(3.8) become

$$\Omega_0 = \frac{\eta_0}{3r_0^3} \tag{5.21}$$

$$n_0 = \tfrac{1}{3}\eta_0 \frac{q}{m}\left(\frac{1}{r_0^3} + \frac{1}{2}\lambda R_0\right) \tag{5.22}$$

$$\eta_0 = -4r_0^3(\omega - \Omega_0)(\kappa + S)\frac{A_0 - 1}{A_0\kappa} \tag{5.23}$$

$$\tfrac{1}{3}A_0^2\eta_0\zeta\left[-\frac{1}{r_0^2} + \frac{1}{2}\lambda(r^2R)'_{r=r_0}\right] - 2A_0r_0n_0 = 2q(\omega - \Omega_0) \tag{5.24}$$

where η_1 has been eliminated in favor of λ using (3.9). Note [from (3.2)] that

$$\kappa = m + O(\zeta^2)$$

and

$$S = (m/4A_0)(1 - A_0) + O(\zeta^2) \tag{5.25}$$

Eliminating Ω_0 from equations (5.21) and (5.23) and substituting for κ and S from (5.25), we find

$$\eta_0 = 3m\omega r_0^2 \frac{r_0}{m}(1+3A_0)\frac{1-A_0}{2A_0+1} \tag{5.26}$$

Equations (5.22) and (5.24) may be solved simultaneously to determine λ, using (5.23) to eliminate the term in $(\omega-\Omega_0)$. After some manipulation one finds

$$\lambda = \frac{4(1+2A_0)}{r_0^3(1+3A_0)(2R_0A_0-2R_0+r_0R_0'A_0)} \tag{5.27}$$

Alternatively, one could obtain equation (5.27) directly from the general expression for λ [equation (3.10)] in the limit $\zeta \ll 1$.

As stated in Section 3, the g-factor is given by $g=2+\lambda$. Equations (3.12) and (5.26) yield

$$J = \frac{1}{2} m\omega r_0^2 \frac{r_0}{m}(1+3A_0)\frac{1-A_0}{2A_0+1} \tag{5.28}$$

Equations (3.15) and (3.16) yield the following expression for the magnetic moment:

$$\mu = \tfrac{1}{2}\omega q r_0^2 \frac{r_0}{m}(1-A_0)\left[\frac{1+3A_0}{2A_0+1}+\frac{2}{r_0^3(2R_0A_0-2R_0+r_0R_0'A_0)}\right] \tag{5.29}$$

5.3. $A_0 \to 1$: The Weak-Field Limit

If $m/r_0 \to 0$ and $q/r_0 \to 0$ with ζ arbitrary but fixed, then $A_0 \to 1$ and space outside (as well as inside) the shell is flat. For this case one would expect to recover the non-general relativistic expressions for the angular momentum, magnetic moment, and g-factor of the shell. In this section we verify that this is so. We proceed (as before) by using the formulas of Section 4 to obtain expressions for μ, J, and g in this limit. From equations (4.3) we find

$$\begin{aligned} C_1 &\sim 1+O(x) \\ C_2 &\sim -2+O(x) \\ C_3 &\sim 3/x+O(x^0) \\ C_4 &\sim -3+O(x) \end{aligned} \tag{5.30}$$

where we have used $(\kappa+S)/r_0 \sim 2x + O(x^2)$ and the C_i have been calculated only to lowest order in x. Equations (4.4) and (4.5) yield

$$\eta_0 = 4m\omega r_0^2 + O(x) \tag{5.31}$$

$$\eta_1 = -\tfrac{2}{3}\omega q r_0^2 + O(x) \tag{5.32}$$

From equations (3.12) and (5.31) we obtain

$$\lim_{A_0 \to 1} J = \tfrac{2}{3} m\omega r_0^2 \tag{5.33}$$

while (4.6), (5.31), and (5.32) give

$$\lim_{A_0 \to 1} \mu = \tfrac{1}{3} q\omega r_0^2 \tag{5.34}$$

From (3.15), (5.33), and (5.34) we find

$$\lim_{A_0 \to 1} g = 1 \tag{5.35}$$

Using Newtonian mechanics, one finds by a simple calculation that the moment of inertia of a rigid spherical shell of mass m and radius r_0 about its axis of rotation is $2mr_0^2/3$ and hence the orbital angular momentum of the shell about its axis of rotation is $2m\omega r_0^2/3$. The angular momentum of the electromagnetic field and the angular momentum due to the stress supporting the shell are, by comparision, $O(q^2/mr_0)$ smaller than the orbital angular momentum, and are therefore negligible (since $q/r_0 \to 0$ in this limit). Thus, the total angular momentum of the shell about its axis of rotation is $2m\omega r_0^2/3$, which is in agreement with (5.33), as one would expect. Similarly, using classical electromagnetism in flat space, one finds that the magnitude of the magnetic moment of such a shell is $q\omega r_0^2/3$, which is in agreement with (5.34), again as one would expect. Given these results for μ and J, it follows that $g = 1$, in agreement with (5.35).

5.4. $\zeta < 1$ and $A_0 \to 0$: The Horizon Radius

Equation (3.1) can be expressed in the form

$$A_0^2 = 1 - 4x + 4\zeta^2 x^2 \tag{5.36}$$

with $x = m/2r_0$ and $\zeta = |q|/m$. For $\zeta < 1$ the rhs of equation (5.36) vanishes when

$$1 - \zeta^2 x = 1/(4x) \tag{5.37}$$

There are two positive values of x, both of which are roots of equation (5.37); it will be shown in Section 6 that only the larger root is physically permissible. For a fixed mass the shell radius approaches a minimum (horizon) value as A_0 approaches zero. We once again use the results of Section 4 to obtain expressions for μ, J, and g as $A_0 \to 0$. For $A_0 \ll 1$, equations (4.3) reduce to

$$\begin{aligned} C_1 &\sim \tfrac{4}{3}(1-\zeta^2 x)(\kappa+S)r_0^{-1} \\ C_2 &\sim 8(r_0^3 Q_0)(\kappa+S) \\ C_3 &\sim 4(1-\zeta^2 x) \\ C_4 &\sim K - 16\zeta^2 x^2 (r_0^4 Q_0) \end{aligned} \tag{5.38}$$

where $K \equiv A_0^2 r_0^4 R_0'$. Note that K approaches a finite, but nonzero value as $A_0 \to 0$, while $Q_0 \to \infty$ and $(\kappa+S)r_0^{-1} \to \infty$ as $A_0 \to 0$. It follows from equations (4.4) and (4.5) that

$$\lim_{A_0 \to 0} \eta_0 = 6m\omega r_0^2 \tag{5.39}$$

and

$$\lim_{A_0 \to 0} \eta_1 = 0 \tag{5.40}$$

where equation (5.37) has been used in (5.39). It follows from equations (3.12) and (5.39) that

$$\lim_{A_0 \to 0} J = m\omega r_0^2 \tag{5.41}$$

and from equations (4.6), (5.39), and (5.40) that

$$\lim_{A_0 \to 0} \mu = q\omega r_0^2 \tag{5.42}$$

Equations (3.15), (5.41), and (5.42) then give

$$\lim_{A_0 \to 0} g = 2 \tag{5.43}$$

6. REMARKS

6.1. Restrictions on m/r_0

Given a value of ζ greater than unity, the minimum value of A^2 is always positive. However, for $\zeta < 1$ the requirement that A^2 be positive implies that either

$$m/r > \zeta^{-2}(1+b) \qquad \forall r \geq r_0$$

or

$$m/r < \zeta^{-2}(1-b) \qquad \forall r \geq r_0$$

The first possibility must be rejected, since it cannot hold for arbitrarily large r. Hence,

$$m/r<\zeta^{-2}(1-b) \qquad \forall r\geq r_0 \tag{6.1}$$

In particular,

$$m/r_0<\zeta^{-2}(1-b), \qquad \zeta<1 \tag{6.2}$$

Note that the horizon radius $r_H=m\zeta^2/(1-b)$ follows from (6.1). In this paper we have not applied any further restrictions on m/r_0, given ζ, other than that stated above. In particular, it is of interest to look at solutions for μ, J, and g as functions of ζ and r_0/m, without restrictions on the sign of κ, which is not observable. (It may be noted that κ is positive for $m/r_0<2\zeta^{-2}$.)

6.2. General Comments on *g*, *μ*, and *J*

From the graphs of μ and J it can be seen that these observables may be either positive or negative (with respect to the sense of ω) for $\zeta>1$. In particular, for certain values of ζ and r_0/m (e.g., Fig. 6b: $\zeta=1.1$, $r_0/m\approx 0.7$) the shell may rotate and yet have no net angular momentum ($J=0$). This situation arises when the (negative) angular momentum associated with the stress supporting the shell exactly cancels the (positive) angular momentum of the rigid-body rotation and the electromagnetic field. In such circumstances the g-factor diverges to infinity, since μ does not in general vanish at the same point as J.

It is of interest to note that for a given ζ (>1) there are some values of g for which there are two possible values of r_0/m (e.g., Fig. 6a: $\zeta=1.1$ and $1\leq g\leq 2$). In these cases one configuration has μ and J both positive, and the other has μ and J both negative. However, since ω and r_0 may not be directly measurable (e.g., for a microscopic system), one cannot in general distinguish between the two configurations by measurements of μ and J.

6.3. Comments on Asymptotic Results

In Section 5 expressions for μ, J, and g were found in four limiting cases. Here we compare results for these cases.

For the two cases $\zeta\gg 1$, $|q|/r_0\ll 1$, and $m/r_0\ll 1$ [Section 5.1(i)] and $\zeta\ll 1$ (Section 5.2) one observes that in the weak-field limit ($x\to 0$ and $A_0\to 1$) the expressions for the angular momentum (5.11) and (5.28) and those for the magnetic moment (5.12) and (5.29) reduce to the non-general relativistic limit values of these quantities for arbitrary ζ, as calculated in Section 5.3. Our results are thus consistent in the Newtonian limit.

For the point particle limit, $r_0/m\sim 0$, and $\zeta\gg 1$ [Section 5.1(ii)] we have already noted from equations (5.17) and (5.18) that both μ and J

vanish as r_0 approaches zero. Thus, such a particle has no external magnetic field, magnetic moment, or angular momentum (both η_1 and η_0 vanish as $r_0 \to 0$). Its electric field is, however, still given by equation (2.8).

For $\zeta < 1$, the shell has an event horizon. As the shell approaches its event horizon, $r_0 = r_H$ (Section 5.4), the ratio of the general relativistic to the nonrelativistic expression for J increases by 50% over its value in the weak-field limit, while the corresponding ratio for μ increases by 200% over its value in the weak-field limit. These results follow from a comparison of equation (5.41) with (5.33) and equation (5.42) with (5.34).

6.4. *g*-Factor for Some Elementary Particles

Charged elementary particles found in nature have $\zeta \gg 1$. It is of interest to see what radii are predicted for the electron, proton, and muon by the asymptotic solution for $\zeta \gg 1$, given the measured g-factors of these particles. [Results for the electron and proton were included in Briggs *et al.* (1981).]

For the electron ($\zeta = 2.042 \times 10^{21}$ in dimensionless units) a value of $g = 2.00232$ yields a shell radius of 2.346×10^{-13} cm.

For the proton ($\zeta = 1.112 \times 10^{18}$) a value of $g = 5.586$ yields a shell radius of 7.789×10^{-17} cm.

For the muon ($\zeta = 9.874 \times 10^{18}$) a value of $g = 2.00234$ yields a shell radius of 1.134×10^{-15} cm.

7. CONCLUSION

For a slowly rotating massive shell of charge-to-mass ratio ζ less than unity, the g-factor has a value in the range $g = 1$ (the weak-field limit) to $g = 2$ (r_0 approaches r_H), in agreement with previous results (Briggs *et al.*, 1981; Cohen *et al.*, 1973). From the asymptotic expressions for μ and J and from graphical plots of μ and J, we find that the ratios of the general relativistic expressions to the nonrelativistic expressions for these quantities are maximized as the shell approaches its event horizon, i.e., $A_0 \to 0$.

For a charge-to-mass ratio greater than unity there is no event horizon and also no upper or lower bound on the value g may take. This arises because the net angular momentum of the shell may vanish for $\zeta > 1$, but not for $\zeta < 1$. For $\zeta \gg 1$ (as it is for charged elementary particles) and positive mass density (i.e., $\kappa \geq 0$), the g-factor lies in the range $1 \leq g \leq 6$.

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